A Method of Stability Analysis of Cylindrical Shells under Biaxial Compression

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The paper applies the method of generalized power series to the analysis of stability of circular cylindrical shells under combined radial pressure and axial force. The numbers of half-wavelengths appearing in original formulas are eliminated using the condition of minimum loading, and effective formulas are derived, expressing the critical loadings in terms of material constants and the geometry of the shell only. Hydrostatic pressure is analyzed as a particular case. Inversion of series makes it possible to obtain direct formulas for the necessary wall thickness if the loadings are given. Separation lines between moderate-length and long shells and between the ranges of elastic and inelastic buckling are discussed in detail. The proposed procedure may be applied to other types of loading as well.

Nomenclature

E = Young's modulus
P = axial force (compressive), referred to unit length of the perimeter
R = mean radius of the shell
S = intensity of loading
U = m^2, number of circumferential half-wavelengths squared
V = n^2, number of axial half-wavelengths squared
$\hat{V}$ = value of V corresponding to one axial half-wavelength
h = thickness of the shell
l = length of the shell
m = number of circumferential half-wavelengths at the length of half-perimeter
n = number of axial half-wavelengths at the length of half-perimeter
p = radial external pressure
q = dimensionless intensity of loading
α = parameter determining the contribution of individual loadings
λ = "transversal slenderness" of the shell, proportional to R/h
ν = Poisson's ratio
σ_p = proportional limit of the material at uniaxial compression

1. Introduction

THERE exists an extensive literature concerning the interaction of critical loadings for circular cylindrical shells. The equations of the type

$$f(p_i,g_i,c_i,m,n) = 0 \qquad (1)$$

(where p_i denote individual loadings, g_i dimensions (geometry) of the shell, c_i material constants, m and n number of half-wavelengths at the length of half-perimeter in circumferential and axial directions, respectively), obtained directly from the theoretical analysis, are ineffective unless m and n are eliminated from the condition of minimal intensity of critical loadings. Papkovitch,[13] Windenburg and Trilling,[16] and Ebner[4] proposed to apply here (in some simple cases of loading) the analytical conditions of minimum; they solved the corresponding equations with respect to m and n approximately and substituted into the original equations, thus obtaining effective formulas. The accuracy of these formulas is rather poor and the range of application limited to very thin shells. Some authors, for example Flügge[5] and Batdorf,[1] presented several diagrams based on numerical calculations. In the problems of combined loadings, the difficulties connected with the elimination of m and n increase. Various approximations have been here proposed: let us mention the papers on combined axial and radial pressure by Hsu, Crate, and Schwartz,[11] Mushtari and Sachenkov,[12] Klein,[9] Immerman,[8] Terebushko,[15] Sharman,[14] Laksmikantham and Gerard,[10] and Horton and Durham.[7]

The principal difficulties are connected with the solution of the obtained analytical conditions of minimum with respect to m and n. The present authors proposed to apply here the expansions into generalized power series, namely in the case of radial pressure[19] and in the case of axial pressure.[2] This method led to relatively simple final formulas with improved accuracy; the formulas for necessary thickness of the wall were obtained as well. The curves separating the ranges of moderate-length and of elastic-plastic buckling were also determined.

The present paper is concerned with the application of generalized power series to the analysis of stability of circular cylindrical shells under simultaneous radial and axial pressure (Fig. 1). Hydrostatic pressure is obtained as a particular case. The convergency of the series is getting worse with increasing axial pressure, and the solution is practically valid for this pressure being not too large. The interaction curves, however, are drawn for the whole of the range, making use of

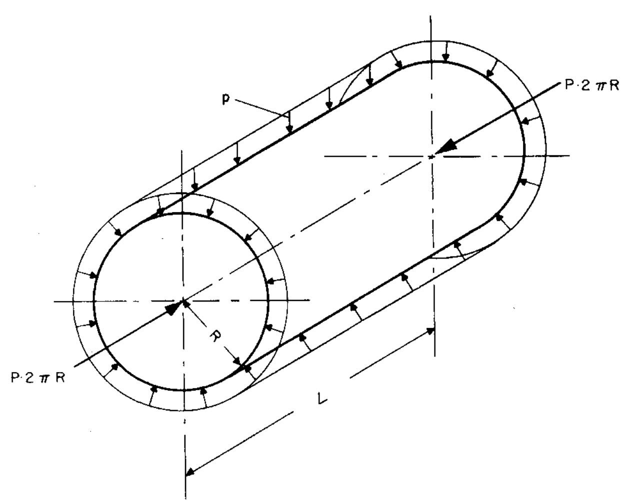

Fig. 1 Combined loading of the shell.

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the solution obtained by Bućko,[2] for the case of pure axial loading.

2. Basic Equations

Our analysis will be based on the formula given for the considered case of simultaneous radial and axial compression of circular cylindrical shell by Girkmann.[6] This formula, determining upper critical pressure for the shell, will be written in the form

$$P_{cr}n^2[(m^2+n^2)^2+(2-\nu)n^2+m^2]+p_{cr}R(m^2+n^2)^2(m^2-1)=Ehn^4+\{Eh^3/[12(1-\nu^2)R^2]\}[(m^2+n^2)^2\times(m^2+n^2-1)^2+2(1-\nu)(n^4-m^4)n^2] \quad (2)$$

where P_{cr} and p_{cr} denote the system of critical loadings (axial force and radial pressure, respectively), E and ν material constants, R and h mean radius and thickness of the shell. Of course, other formulas of the type (2) may be used as basic formulas for the analysis as well; the method is quite general.

We introduce now the critical loading intensity S by the formulas

$$p_{cr}=S\cos\alpha \qquad P_{cr}=SR\sin\alpha \quad (3)$$

where

$$\alpha=\arctan(P_{cr}/p_{cr}R) \quad (4)$$

is a dimensionless parameter characterizing the ratio of external loadings (axial force to radial pressure). Introduce, furthermore, the following dimensionless quantities, similarly as in Refs. 2, 19, and 20:

$$q=\{[12(1-\nu^2)R^3]/Eh^3\}S \quad (5)$$

which is the dimensionless intensity of critical loadings, and

$$\lambda=[12(1-\nu^2)]^{1/2}(R/h) \quad (6)$$

which is the dimensionless "transversal slenderness," and rewrite (2) in the form

$$q=\frac{V^2\lambda^2+(U+V)^2(U+V-1)^2+2(1-\nu)(V^2-U^2)V}{V[(U+V)^2+(2-\nu)V+U]\sin\alpha+(U+V)^2(U-1)\cos\alpha} \quad (7)$$

where $U=m^2$, $V=n^2$. The definition of λ, (6), resembles that of the slenderness of a column with rectangular cross section.

The third term in the numerator is usually very small with respect to the remaining terms and will be neglected without lack of accuracy.

At first we are going to eliminate from this formula the numbers of half-wavelengths, i.e., U and V. They should be chosen in such a way as to assure minimal critical load intensity q. It turns out that the derivative $\partial q/\partial V>0$ (for each loading acting separately, this was stated in Refs. 2 and 19), and we have to assume the minimal value of V. It corresponds to one axial half-wave at the length l of the shell (greater number of half-waves at pure axial compression is observed just at finite deflections) and will be denoted by $\hat{V}$:

$$V=\hat{V}=\pi^2(R/l)^2 \quad (8)$$

The value of U is found from the analytical condition of minimum:

$$\partial q/\partial U=0 \quad (9)$$

Actually U has to be an integer squared, but the assumption (9) causes small errors increasing the safety and leads to uniform results, namely to one formula for the entire range.

3. Determination of Critical Load Intensity for Moderate-Length Shells

Performing the differentiation (9), we arrive at the equation

$$2[2(U+\hat{V})^3-3(U+\hat{V})^2+(U+\hat{V})]\{\hat{V}[(U+\hat{V})^2+(2-\nu)\hat{V}+U]\sin\alpha+(U+\hat{V})^2(U-1)\cos\alpha\}-[\hat{V}^2\lambda^2+(U+\hat{V})^4-2(U+\hat{V})^3+(U+\hat{V})^2]\{[2\hat{V}(U+\hat{V})+\hat{V}]\sin\alpha+(U+\hat{V})(3U+\hat{V}-2)\cos\alpha\}=0 \quad (10)$$

This is an algebraic equation of the sixth degree with respect to U. To solve it, we use generalized power series of the parameter λ (transversal slenderness); assume, namely, the solution in the form

$$U=\sum_{j=0}^{\infty}a_j\lambda^{\mu\times\psi_j} \quad (11)$$

it turns out that the constants are equal to $\mu=\frac{1}{2}$, $\psi=-\frac{1}{2}$. Because of negative value of the constant ψ in this series, it will be rapidly convergent for very thin shells, since λ for such shells is very large. Substituting (11) into (10) and using the method of equal coefficients,[17,18] we arrive finally at

$$U=(3\hat{V}^2)^{1/4}\lambda^{1/2}+\frac{1-2\hat{V}-\hat{V}\tan\alpha}{3}+\frac{14\hat{V}^2+10\hat{V}+5-(140\hat{V}^2+34\hat{V})\tan\alpha-42\hat{V}^2\tan^2\alpha}{36\,(3\hat{V}^2)^{1/4}}\times\lambda^{-1/2}+\ldots \quad (12)$$

This series, as well as the following ones, may be used only in a certain interval of the parameter α. For $\alpha=\pi/2$ ($p_{cr}=0$, pure axial compression), this series is without any sense, since $\tan\alpha=\infty$; as a matter of fact, the corresponding series derived in Ref. 2 for the case of pure compression is of a somewhat different form. But even for α close to $\pi/2$, the series is divergent. Thus we confine ourselves to the shells loaded by high radial pressure and relatively small axial force; the interval $\tan\alpha<3$ seems to deliver reasonable limitation. In any case, the hydrostatic pressure will clearly lie inside the limits of applicability of the obtained formulas. These formulas may be regarded as a generalization of those derived in Ref. 18 for pure radial pressure.

Substituting U determined by (12) into formula (7), we obtain the final formula for the critical intensity:

$$q_{\min}=(3\cos\alpha)^{-1}\{4(3\hat{V}^2)^{1/4}\lambda^{1/2}+[2(2\hat{V}-1)-4\hat{V}\tan\alpha]+([1+2\hat{V}+10\hat{V}^2-(14\hat{V}+20\hat{V}^2)\tan\alpha+10\hat{V}^2\tan^2\alpha]/3(3\hat{V}^2)^{1/4})\lambda^{-1/2}+\ldots\} \quad (13)$$

Making use of (3), we may determine now both critical pressures in terms of the parameter α: critical radial pressure is

$$p_{cr}=\{E[12(1-\nu^2)]^{1/2}/3\}\{4(3\hat{V}^2)^{1/4}\lambda^{-5/2}+[2(2\hat{V}-1)-4\hat{V}\tan\alpha]\lambda^{-3}+([1+2\hat{V}+10\hat{V}^2-(14\hat{V}+20\hat{V}^2)\tan\alpha+10\hat{V}^2\tan^2\alpha]/3(3\hat{V}^2)^{1/4})\lambda^{-7/2}+\ldots\} \quad (14)$$

and critical axial loading is

$$P_{cr}=\{E[12(1-\nu^2)]^{1/2}/3\}\tan\alpha\{4(3\hat{V}^2)^{1/4}\lambda^{-5/2}+[2(2\hat{V}-1)-4\hat{V}\tan\alpha]\lambda^{-3}+([1+2\hat{V}+10\hat{V}^2-(14\hat{V}+20\hat{V}^2)\tan\alpha+10\hat{V}^2\tan^2\alpha]/3(3\hat{V}^2)^{1/4})\lambda^{-7/2}+\ldots\} \quad (15)$$

Substituting here $\alpha=0$, we obtain for p_{cr} the formula derived in Ref. 19 and $P_{cr}=0$. For the important case of hydrostatic pressure acting on a closed cylinder, we have

$$p_{cr} = P_{cr}R/2 \qquad \tan\alpha = \tfrac{1}{2} \tag{16}$$

$$U = (3\hat{V}^2)^{1/4}\lambda^{1/2} + \frac{2 - 5\hat{V}}{6} + \frac{10 - 14\hat{V} - 133\hat{V}^2}{72(3\hat{V}^2)^{1/4}}\lambda^{-1/2} + \ldots \tag{17}$$

$$p_{cr} = \frac{E[12(1-\nu^2)]^{1/2}}{3}\left[4(3\hat{V}^2)^{1/4}\lambda^{-5/2} + 2(\hat{V} - 1)\lambda^{-3} + \frac{2 - 10\hat{V} + 5\hat{V}^2}{6(3\hat{V}^2)^{1/4}}\lambda^{-7/2} + \ldots\right] \tag{18}$$

Equations (14) and (15) may be treated as parametrical equations of the interaction curve for the considered case of stability under combined loadings. For certain engineering applications, it may be important to obtain such an equation in explicit form. This equation might be obtained using directly the formulas derived in Ref. 18 for deparametrization of two generalized power series or subsequently performing the inversion of (15) and substitution into (14). It turns out, however, that the series (15) in its present form cannot be inverted with respect to $\tan\alpha$: we obtain the same powers of λ in all terms of the series. At first we have to introduce a new loading parameter, namely,

$$\bar{P}_{cr} = (P_{cr}/ER)\lambda^{5/2} \tag{19}$$

thus confining to smaller values of P_{cr}. Then the series may be inverted with respect to $\tan\alpha$ and substituted into (14); finally,

$$p_{cr} = E[12(1-\nu^2)]^{1/2}\left\{\frac{4}{3}(3\hat{V}^2)^{1/4}\lambda^{-5/2} + \left[\frac{2}{3}(2\hat{V} - 1) - \frac{\hat{V}}{2(27\hat{V}^2)^{1/4}(1-\nu^2)^{1/2}}\bar{P}_{cr}\right]\lambda^{-3} + \left[\frac{1 + 2\hat{V} + 10\hat{V}^2}{9(3\hat{V}^2)^{1/4}} - \frac{3(3\hat{V}^2)^{1/4} + 14 + 20\hat{V}}{18(1-\nu^2)^{1/2}}\bar{P}_{cr} - \frac{6\hat{V} - 5}{96(1-\nu^2)(27\hat{V}^2)^{1/4}}\bar{P}_{cr}^2\right]\lambda^{-7/2} + \ldots\right\} \tag{20}$$

4. Long Shells

The number of circumferential half-wavelengths is determined by (12) if $U = m^2 > 4$. For longer shells, we have to assume that $U = 4$, since this value corresponds to the minimal really possible number of half-wavelengths (excluding buckling of Euler's type, $m = 1$). Such shells will be called long shells, similarly as in Ref. 19. Substituting $U = 4$ and $V = \hat{V}$ into (7), we obtain finally

$$p_{cr} = \frac{E[12(1-\nu^2)]^{1/2}}{\lambda^3} \times \frac{\hat{V}^2\lambda^2 + (4+\hat{V})^2(3+\hat{V})^2}{\hat{V}[(4+\hat{V})^2 + (2-\nu)\hat{V} + 4]\tan\alpha + 3(4+\hat{V})^2} \tag{21}$$

$$P_{cr} = \frac{ER[12(1-\nu^2)]^{1/2}}{\lambda^3} \times \frac{\hat{V}^2\lambda^2 + (4+\hat{V})^2(3+\hat{V})^2}{\hat{V}[(4+\hat{V})^2 + (2-\nu)\hat{V} + 4]\tan\alpha + 3(4+\hat{V})^2}\tan\alpha \tag{22}$$

Elimination of α leads here to the explicit equation of the interaction curve:

$$p_{cr} = \frac{E[12(1-\nu^2)]^{1/2}}{\lambda^3}\,\frac{\hat{V}^2\lambda^2 + (4+\hat{V})^2(3+\hat{V})^2}{3(4+\hat{V})^2} - \frac{\hat{V}[(4+\hat{V})^2 + (2-\nu)\hat{V} + 4]}{3(4+\hat{V})^2}\,\frac{P_{cr}}{R} \tag{23}$$

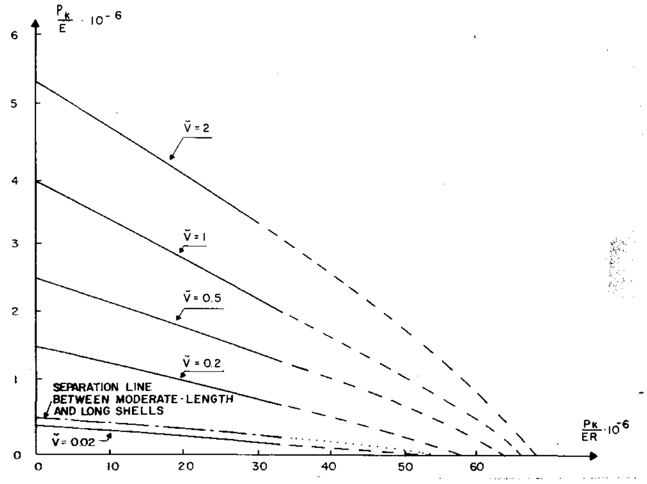

Fig. 2 Interaction curves in the plane axial force; radial pressure.

Hence we have arrived here at a segment of a straight line in the P_{cr}-p_{cr} plane.

The interaction curves (2) and (23) are shown in Fig. 2 for $\lambda = 300$.

5. Direct Design of Shells with Respect to Their Stability

The derived formulas determine critical loadings in terms of the given geometry of the shell. In the present section, we invert the series as to obtain direct formulas for the required shell thickness h in terms of the loadings p_{cr} and P_{cr} (which should be equal to the actual loadings multiplied by a suitable safety factor) and of the dimensions R and l. Such formulas are particularly convenient in engineering applications.

The inversion of the series (14) with respect to λ yields for moderate-length shells

$$\frac{h}{R} = \left[\frac{729(1-\nu^2)^3}{4\hat{V}^2}\right]^{1/10}\left(\frac{P_{cr}}{E}\right)^{2/5} - \left[\frac{9(1-\nu^2)}{16\hat{V}^4}\right]^{1/5} \times \frac{2\hat{V} - 1 - 2\hat{V}\tan\alpha}{5}\left(\frac{P_{cr}}{E}\right)^{3/5} + \left[\frac{1-\nu^2}{48\hat{V}^4}\right]^{1/10} \times \frac{17 - 128\hat{V} + 8\hat{V}^2 + (248\hat{V} - 16\hat{V}^2)\tan\alpha + 8\hat{V}^2\tan^2\alpha}{200\hat{V}} \times \left(\frac{P_{cr}}{E}\right)^{4/5} + \ldots \tag{24}$$

Assuming, for example, $\tan\alpha = \frac{1}{2}$, we obtain for a shell under hydrostatic pressure

$$\frac{h}{R} = \left[\frac{729(1-\nu^2)^3}{4\hat{V}^2}\right]^{1/10}\left(\frac{p_{cr}}{E}\right)^{2/5} - \left[\frac{9(1-\nu^2)}{16\hat{V}^4}\right]^{1/5}\frac{\hat{V} - 1}{5}\left(\frac{p_{cr}}{E}\right)^{3/5} + \left[\frac{1-\nu^2}{48\hat{V}^4}\right]^{1/10} \times \frac{17 - 4\hat{V} + 2\hat{V}^2}{200\hat{V}}\left(\frac{p_{cr}}{E}\right)^{4/5} + \ldots \tag{25}$$

Consider now long shells; the required ratio h/R may be here determined by a closed-form formula. At first we write (21) in the form

$$\left(\frac{h}{R}\right)^3 + \frac{12(1-\nu^2)\hat{V}^2}{(4+\hat{V})^2(3+\hat{V})^2}\left(\frac{h}{R}\right) - 12(1-\nu^2) \times \frac{\hat{V}[(4+\hat{V})^2 + (2-\nu)\hat{V} + 4]\tan\alpha + 3(4+\hat{V})^2}{(4+\hat{V})^2(3+\hat{V})^2}\,\frac{p_{cr}}{E} = 0 \tag{26}$$

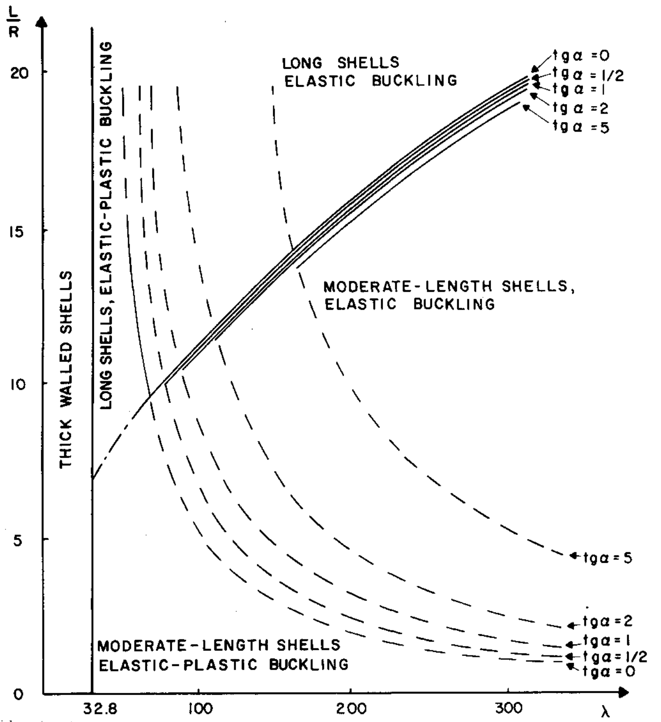

Fig. 3 Limit curves in the plane transversal slenderness; length of the shell.

This cubic equation with respect to h/R is already in its canonical form. The discriminant is here positive; hence

$$\frac{h}{R} = \left\{\frac{W_2}{2}\frac{p_{cr}}{E} + \left[\frac{W_2^2}{4}\frac{p_{cr}^2}{E^2} + \frac{W_1^3}{27}\right]^{1/2}\right\}^{1/3} + \left\{\frac{W_2}{2}\frac{p_{cr}}{E} - \left[\frac{W_2^2}{4}\frac{p_{cr}^2}{E^2} + \frac{W_1^3}{27}\right]^{1/2}\right\}^{1/3} \quad (27)$$

where

$$W_1 = \frac{12(1-\nu^2)\hat{V}^2}{(4+\hat{V})^2(3+\hat{V}^2)} \quad (28a)$$

$$W_2 = \frac{\hat{V}[(4+\hat{V})^2 + (2-\nu)\hat{V} + 4]\tan\alpha + 3(4+\hat{V})^2}{(4+\hat{V})^2(3+\hat{V})^2} \times 12(1-\nu^2) \quad (28b)$$

6. Separation Line between Long Shells and Moderate-Length Shells

Substituting $U = 4$ into Eq. (10), we determine the separation line (in the plane V-λ) between long shells and moderate-length shells. Using generalized power series to solve this equation (of the sixth degree) with respect to the boundary value V_b, we arrive finally at

$$\hat{V}_b = \frac{12(10)^{1/2}}{5}\lambda^{-1} + \frac{468 + 483\tan\alpha}{25}\lambda^{-2} + \ldots \quad (29)$$

This formula enables us to determine the boundary length of the shell of given dimensions. For $\hat{V} < \hat{V}_b$, i.e., $l > l_b$, we have to assume $U = 4$; it means that the shell must be considered as a long one. The dependence of $\hat{V}_b$ on α (ratio of loadings) is rather insignificant. The line (29) is shown in Fig. 3.

In the problem of direct design of shells, the formula (29) is inapplicable, since the transversal slenderness λ is not known. Eliminating λ with the aid of (21), namely, looking for the solution of (21) with respect to λ in the form of a generalized power series, we find here

$$\hat{V}_b = 4\left[\frac{9(10)^{1/2}}{35[3(1-\nu^2)]^{1/2}}\right]^{1/3}\left(\frac{p_{cr}}{E}\right)^{1/3} + \frac{1}{5}\left[\frac{14[3(1-\nu^2)]^{1/2}}{45}\right]^{1/3}\left[708 + 683\tan\alpha - \frac{1712 + 322\tan\alpha}{7\,[3(1-\nu^2)]^{1/2}}\right]\left(\frac{p_{cr}}{E}\right)^{2/3} + \ldots \quad (30)$$

Even here the dependence of V_b on α is insignificant.

7. Limitation of the Derived Formulas with Respect to the Assumption of Elastic Buckling

All of the derived formulas are valid only for elastic buckling of the shell. The problem of inelastic buckling was treated in a similar way by Bućko[3] but with the restriction to pure radial pressure.

Here we are going to determine only the range of validity of our present considerations. The shell is subject to plane stress, determined by principal stresses σ_1 and σ_2, where

$$\sigma_1 = P_{cr}/h \qquad \text{(axial stress)}$$
$$\sigma_2 = P_{cr}R/h \qquad \text{(circumferential stress)} \quad (31)$$

Using the Huber-Mises-Hencky strength theory, we write down the condition of elastic range in the form

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 \leqslant \sigma_p^2 \quad (32)$$

σ_p being the proportional limit of the material at uniaxial compression. Substituting (31), we arrive at

$$\left(\frac{P_{cr}}{h}\right)^2 + \left(\frac{p_{cr}R}{h}\right)^2 - \frac{p_{cr}P_{cr}R}{h^2} \leqslant \sigma_p^2 \quad (33)$$

We have now to consider moderate-length shells and long shells separately.

7.1. Moderate-Length Shells

To determine the boundary slenderness λ, we substitute (14) and (15) into (33). Denote this boundary slenderness by λ_b, and replace the sign of inequality by the sign of equality. Performing the operations on power series and inverting (33) with respect to $\lambda = \lambda_b$, we obtain finally

$$\lambda_b = \left[\frac{256\hat{V}^2(1+\tan\alpha+\tan^2\alpha)^2}{27}\right]^{1/6}\left(\frac{\sigma_p}{E}\right)^{-2/3} + \left[\frac{16(1+\tan\alpha+\tan^2\alpha)}{19{,}683}\right]^{1/2}(2\hat{V} - 1 - 2\hat{V}\tan\alpha)\times\left(\frac{\sigma_p}{E}\right)^{-1/3} + \ldots \quad (34)$$

For $\lambda \geqslant \lambda_b$, we have elastic buckling, and the derived formulas are valid.

In the problem of direct design of shells, we have to eliminate the unknown here quantity λ. Substituting (34) into (14), we eliminate λ and obtain

$$\left(\frac{p_{cr}}{E}\right)_b = \frac{3(1-\nu^2)^{1/2}}{(2\hat{V})^{1/3}}(1+\tan\alpha+\tan^2\alpha)^{-5/6}\left(\frac{\sigma_p}{E}\right)^{5/3} - \frac{(1-\nu^2)^{1/2}}{2\hat{V}}\frac{2\hat{V} - 1 - 2\hat{V}\tan\alpha}{1+\tan\alpha+\tan^2\alpha}\left(\frac{\sigma_p}{E}\right)^2 + \ldots \quad (35)$$

The buckling of the shell is elastic for p_{cr}/E less than the boundary value (35).

7.2. Long Shells

For long shells, we obtain the formulas in a closed form. Substituting (21) and (22) and solving with respect to λ, we arrive at

$$\lambda_b = (4 + \hat{V})(3 + \hat{V})(1 + \tan\alpha + \tan^2\alpha)^{1/4} \times [\{\hat{V}[(4 + \hat{V})^2 + (2 - \nu)\hat{V} + 4]\tan\alpha + 3(4 + \hat{V})^2\}(\sigma_p/E) - \hat{V}^2(1 + \tan\alpha + \tan^2\alpha)^{1/2}]^{-1/2} \quad (36)$$

Boundary value of the critical radial pressure p_{cr} may be obtained by the substitution of (36) into (21):

$$(p_{cr}/E)_b = [\{\hat{V}[(4 + \hat{V})^2 + (2 - \nu)\hat{V} + 4]\tan\alpha + 3(4 + \hat{V})^2\}(\sigma_p/E) - \hat{V}^2(1 + \tan\alpha + \tan^2\alpha)^{1/2}]^{1/2} \times [12(1 - \nu^2)]^{1/2}(\sigma_p/E)(4 + \hat{V})^{-1}(3 + \hat{V})^{-1}(1 + \tan\alpha + \tan^2\alpha)^{-1/4} \quad (37)$$

Similarly, we can derive the formula for the boundary value of the critical intensity of loadings, S_b.

8. Summary of Results

The obtained regions of applicability of the derived formulas are shown in Fig. 3 for $\sigma_p/E = 0.001$, $\nu = 0.3$, with $\tan\alpha$ as a parameter. The insignificant dependences on α are seen. Similar diagrams may be obtained for the problem of direct design of shells.

9. Final Remarks

The derived formulas solve the problem of elastic buckling of a cylindrical shell under simultaneous radial pressure and axial force. The parameter α, corresponding to the ratio of pressures, is theoretically arbitrary, but practically must not be too high to assure the convergence of the series. The region $\tan\alpha \approx \infty$ (small radial pressure superposed on axial force) must be treated separately.

The paper is based on the linear theory of shell buckling; thus, it refers to the upper critical pressure. However, in the considered range of loadings, the differences with respect to the lower critical pressure, determined by the nonlinear theory, are not very large, and for most applications this fact may be taken into account by means of a suitably chosen safety factor.

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